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ON GRADED SEMI-PRIME RINGS

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ABSTRACT. Let G be a group with identity e and let R be a G-graded ring . In this article, we study the concept of graded semi-prime rings and we give some results on this concept. For example, we prove that if R is graded semi-prime and I is a graded ideal of R, then the set of all left annihilators of I equals the set of all right annihilators of I. Also, we prove that certain subsets should lie in the center of the graded semi-prime ring R.

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1. INTRODUCTION

Let G be a group with identity e and R be a ring with unity 1. Then R is called a G-graded (or gr-ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and all the homogeneous elements of R are denoted by $h(R)$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Let I be an ideal of R. Then I is called G-graded ideal if $I = \bigoplus_{g \in G}(I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g \in G} x_g$, then $x_g \in I$ for all $g \in G$. Left and right G-graded ideals are defined analogously. An ideal of a G-graded ring need not be G-graded. To see this, consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Now, $I = \langle 1 + i \rangle$ is an ideal of R with $1 + i \in I$. If I is G-graded, then $1 \in I$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbf{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not G-graded. For more details, one can look in [2], [3] and [4].

2. GRADED SEMI-PRIME RINGS

A gr-ring R is said to be graded prime (gr-prime) if whenever $I \neq \{0\}$ and $J \neq \{0\}$ are graded ideals of R, then $IJ \neq \{0\}$ [1]. As is trivial, the definition above is equivalent to the statement: if $aRb = \{0\}$, then either $a = 0$ or $b = 0$ where $a, b \in h(R)$. Also, this is equivalent to the statement: A gr-ring R is gr-prime if and only if the right annihilator of a nonzero gr-right ideal of R should be $\{0\}$. It follows that if $I \neq \{0\}$ is a gr-left ideal and $J \neq \{0\}$ is a gr-right ideal in a gr-prime ring R, then $I \bigcap J \neq \{0\}$.

A gr-ring R is said to be graded semi-prime (gr-semi-prime) if it has no nonzero nilpotent gr-ideals. Graded semi-primeness like gr-primeness, can be characterized in terms of elements of the ring: if $aRa = \{0\}$ where $a \in h(R)$, then $a = 0$. We begin with an easy and useful result.

Lemma 2.1. *If R is a gr-prime ring with no nonzero nilpotent homogeneous elements, then R has no homogeneous zero divisors.*

Proof. Let $a, b \in h(R)$ such that $ab = 0$. Then $(ba)^2 = (ba)(ba) = b(ab)a = 0$ and then by assumption $ba = 0$. Let $x \in R$. Then $x = \sum_{g\in G} x_g$. For any $g \in G$, $(ax_gb)^2 = (ax_gb)(ax_gb) = ax_g(ba)x_gb = 0$ and then $(ax_gb) = 0$. So, $0 = \sum_{g\in G} ax_gb = a(\sum_{g\in G} x_g)b = axb$. Hence, $aRb = \{0\}$. Since R is gr-prime, either $a = 0$ or $b = 0$. □

For gr-semi-prime rings, we have an ideal analog of Lemma 2.1:

Lemma 2.2. *If R is a gr-semi-prime ring and I, J are gr-ideals of R such that $IJ = \{0\}$, then $JI = \{0\}$.*

Proof. Since $IJ = \{0\}$, $(JI)^2 = (JI).(JI) = J(IJ)I = \{0\}$ and since R is gr-semi-prime, $JI = \{0\}$. □

In fact, we can say a little more. Let X be a subset of R and $r(X) = \{y \in R : xy = 0 \text{ for all } x \in X\}$ and $l(X) = \{y \in R : yx = 0 \text{ for all } x \in X\}$ denote the right and the left annihilators of X, respectively. Clearly, $r(X)$ ($l(X)$) is a right (left) ideal of R. If X is a right (left) ideal, then $r(X)$ is an ideal. In fact, if X is a graded right (left) ideal of a graded ring R, then $r(X)$ ($l(X)$) is a graded ideal.

Lemma 2.3. *Let R be a G-graded ring and X be a graded right ideal of R. Then $r(X)$ is a graded ideal of R.*

Proof. Clearly, $r(X)$ is an ideal of R. Let $y = \sum_{g\in G} y_g \in r(X)$ and x be a homogeneous element of X. Then $xy - \sum_{g\in G} xy_g = 0$. This implies that $xy_g = 0$ for all $g \in G$. Since y_g annihilates every homogeneous element in X and X is generated by homogeneous elements, we have $y_g \in r(X)$ for all $g \in G$. Hence $r(X)$ is a graded ideal of R. □

Similarly, one can prove that if X is a graded left ideal of a graded ring R, then $l(X)$ is a graded ideal of R.

Theorem 2.4. *If R is a gr-semi-prime ring and I is a gr-ideal of R, then $r(I) = l(I)$.*

Proof. Let $J = r(I)I$. Then J is a gr-ideal of R and $J^2 = r(I)I.r(I)I = r(I).(Ir(I)).I = \{0\}$ and then $J = \{0\}$. That is, $r(I)I = \{0\}$ and so $r(I) \subseteq l(I)$. Similarly, $l(I) \subseteq r(I)$, hence we conclude that $r(I) = l(I)$. □

Theorem 2.5. *If R is a gr-semi-prime ring and I is a gr-ideal of R, then $I \bigcap r(I) = \{0\}$.*

Proof. $I \bigcap r(I)$ is a gr-ideal of R and $(I \bigcap r(I))^2 \subseteq Ir(I) = \{0\}$. Therefore, $I \bigcap r(I) = \{0\}$. □

Let R be a G-graded ring and $g \in G$. Then R is said to be g-semi-prime if whenever $a \in R_e$ such that $aR_ga = \{0\}$, then $a = 0$. Every graded domain $R = \bigoplus_{g\in G} R_g$ is a g-semi-prime for $R_g \neq 0$.

Proposition 2.6. *Let R be a G-graded ring and $g \in G$. Suppose R is a g-semi-prime and $a \in R_e$ satisfies $a(ay - ya) = 0$ for all $y \in R_g$. Then $a \in Z(R_e)$ (the center of the subring R_e).*

Proof. Let $x \in R_g$ and $r \in R_e$. Then $a(a(xr) - (xr)a) = 0$. However, $a(xr) - (xr)a = (ax - xa)r + x(ar - ra)$ and then $0 = a(ax - xa)r + ax(ar - ra)$ and since $a(ax - xa) = 0$, $ax(ar - ra) = 0$ for all $x \in R_g$, that is $aR_g(ar - ra) = \{0\}$. But this gives that $(ar - ra)R_g(ar - ra) = \{0\}$. Since R is g-semi-prime, $ar - ra = 0$ for all $r \in R_e$ and hence $a \in Z(R_e)$. □

From Proposition 2.6, we can state a series of results.

Theorem 2.7. *Let R be a gr-prime ring and $a \in h(R)$ centralizes a non-zero gr-right ideal of R. Then $a \in Z(R)$.*

Proof. Let I be the nonzero gr-right ideal of R that is centralized by a. Let $x \in h(R)$ and $r \in I$. Then $rx \in I$ and $a(rx) = (rx)a$. But $ar = ra$, so $r(ax - xa) = 0$ which is $I(ax - xa) = \{0\}$ for all $x \in h(R)$. Since I is nonzero and R is gr-prime, $ax = xa$ for all $x \in h(R)$. Hence, $a \in Z(R)$. □

We now show that in any gr-ring, the annihilator of a large set of commutators is a gr-ideal of that gr-ring.

Lemma 2.8. *Let R be a gr-ring and let $u \in h(R)$. If $V_u = \{a = \sum_{g \in G} a_g \in R : a_g(ux - xu) = 0$ for all $x \in h(R)$ and $g \in G\}$, then V_u is a gr-ideal of R.*

Proof. It is easily checked that if $a, b \in V_u$, then $a - b \in V_u$. It is also clear that V_u is a left ideal of R. We show that V_u is a right ideal of R. Let $a \in V_u$ and $x, r \in h(R)$. Then $a(u(rx) - (rx)u) = 0$. But $u(rx) - (rx)u = (ur - ru)x + r(ux - xu)$. Hence, $0 = a(ur - ru)x + ar(ux - xu)$ that is $ar(ux - xu) = 0$. Hence, $ar \in V_u$ for every $r \in h(R)$. Since V_u is closed under addition, this implies that V_u is a right ideal of R. So V_u is an ideal of R. Clearly, V_u is a graded ideal of R by definition. □

As an immediate consequence we have:

Theorem 2.9. *Let R be a gr-prime ring and $u \in h(R)$. If $V_u \neq 0$, then $u \in Z(R)$.*

Proof. By Lemma 2.8, V_u is a gr-ideal of R. By definition of V_u, $ux - xu \in r(V_u)$ for all $x \in h(R)$. Since R is gr-prime, $V_u r(V_u) = 0$ implies $r(V_u) = 0$ and so $ux = xu$ for all $x \in h(R)$. This implies that $u \in Z(R)$. □

The center of a gr-ring is, after all, the set of homogeneous elements commuting with all homogeneous elements of the gr-ring. For gr-semi-prime rings we can show that centralizing a somewhat smaller part of the gr-ring already forces membership in the center. The result we prove below is one of a large class of results of this nature which can be proved.

Theorem 2.10. *Let R be a gr-semi-prime ring and suppose that $a \in h(R)$ centralizes all commutators $xy - yx$, $x, y \in h(R)$. Then $a \in Z(R)$.*

Proof. Let $x, y \in h(R)$. Then since $x(ya) - (ya)x$ is a commutator, a should commute with $x(ya) - (ya)x$. But $x(ya) - (ya)x = (xy - yx)a + y(xa - ax)$. By assumption, a commutes with the left side and the first term of the right side of this last relation. The net result is that a should commute with $y(xa - ax)$ for all $x, y \in h(R)$. This gives us that $(ya - ay)(xa - ax) = 0$ for all $x, y \in h(R)$. If $V_a = \{r \in h(R) : r(xa - ax) = 0 \text{ for all } x \in h(R)\}$, then by Lemma 2.8, V_a is a gr-ideal of R and by the above, $ya - ay \in V_a$ for all $y \in h(R)$. On the other hand, from the definition of V_a, all $ya - ay \in r(V_a)$ and hence all $ya - ay \in r(V_a) \bigcap V_a$. Since R is gr-semi-prime, $r(V_a) \bigcap V_a = \{0\}$ and hence $ya - ay = 0$ for all $y \in h(R)$, i.e., $a \in Z(R)$. □

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